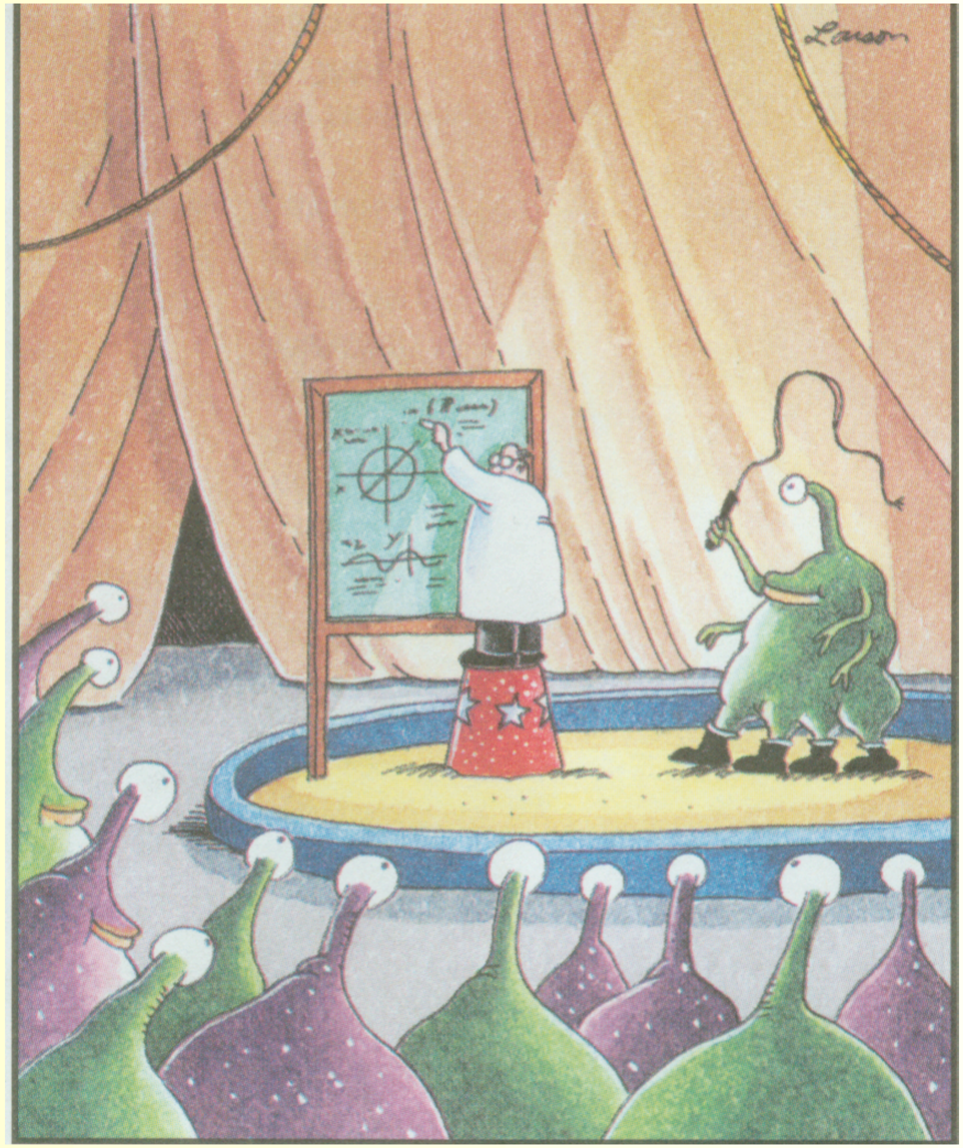
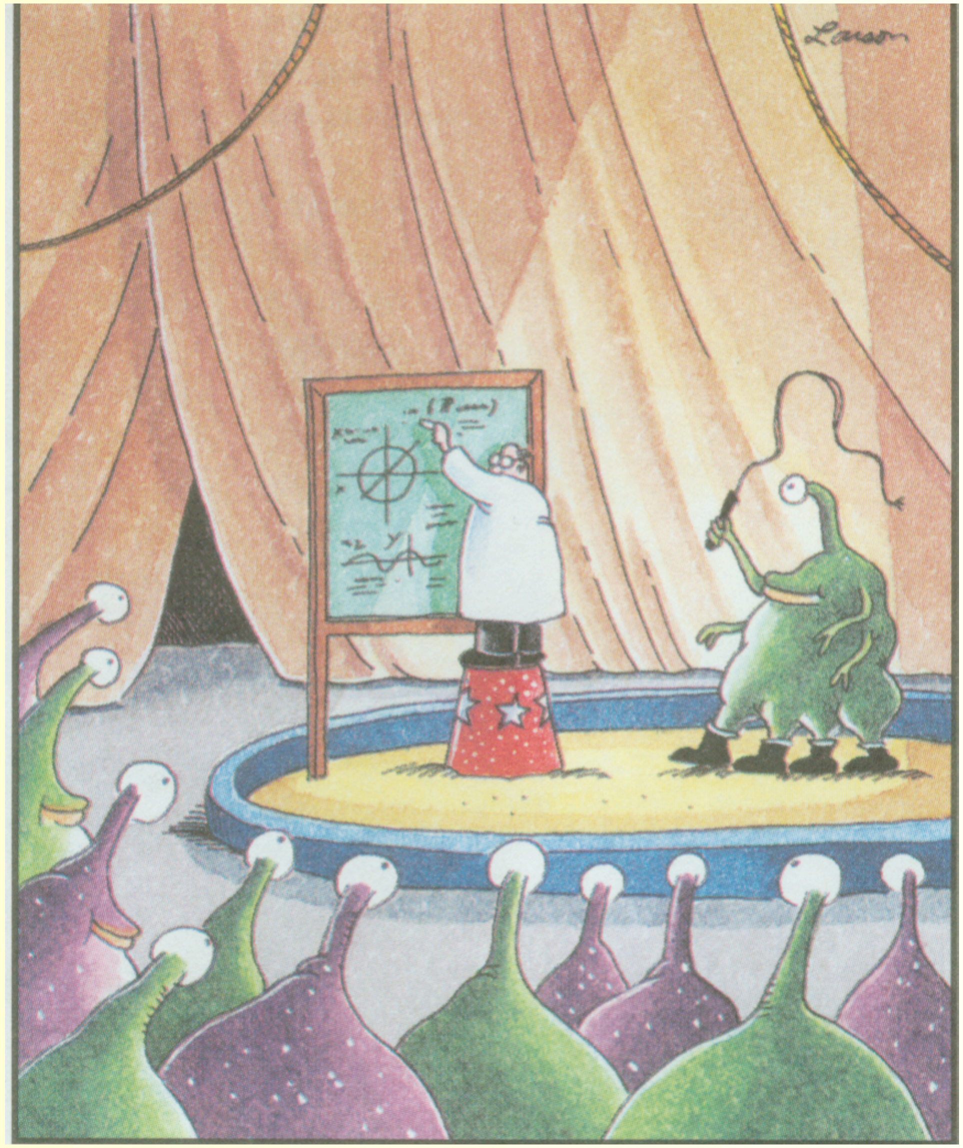
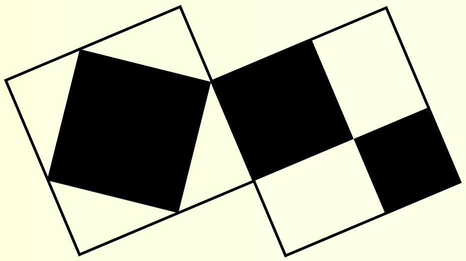

Larson



Abducted by an alien circus company, Professor Doyle is forced to write calculus equations in center ring.

Ideal Multiplication in Fields of Low Degree
A Preliminary Report



Alf van der Poorten

ceNTRe for Number Theory Research, Sydney

To Richard Brent on his 60th birthday

Computing by the Numbers   Berlin, July 21, 2006

Manjul Bhargava's work on higher composition laws amply deals with issues that arise in congenially detailing an algorithm for multiplication of ideals in quadratic fields. I briefly illustrate Bhargava's work by giving my take on its application to the well known quadratic case, hoping thereby to instance and hint at its generalisations to cubic, and quartic fields.

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Composition of Quadratic Forms

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The product of two quadratic forms $\varphi = UX^2 + VXY + WY^2$ and $\varphi' = U'X'^2 + V'X'Y' + W'Y'^2$ is a nasty expression

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Suppose, however, that it happens to happen that there is a bilinear substitution

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Then we have a much more palatable "product" and, moreover, we may then report that the form Φ is a compound of the given forms φ and φ'.

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$$\begin{aligned}&(2X^2+2XY+3Y^2)(2{X'}^2+2XY'+3{Y'}^2)\\&\qquad=(2XX'+XY'+X'Y-2YY')^2+5(XY'+XY'+YY')^2\,.\end{aligned}$$

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$$= (2XX'+XY'+X'Y-2YY')^2+5(XY'+XY'+YY')^2 .$$

In other words, x^2+5y^2 is a compound of $2x^2+2xy+3y^2$ with itself.

If $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, I write $\Phi_M(x, y) = \Phi(ax + by, cx + dy)$, and recall that the discriminant of Φ_M is $(\det M)^2$ times that of Φ.

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Now notice that

$$x = (A_xX'+B_xY')X+(C_xX'+D_xY')Y = (A_xX+C_xY)X'+(B_xX+D_xY)Y'$$

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Hence we have the identities

$$\varphi(X,Y)\varphi'(X',Y') = \Phi(x,y) = \Phi_{\left(\begin{smallmatrix} A_xX'+B_xY' & C_xX'+D_xY' \\ Ax'+By' & Cx'+Dy' \end{smallmatrix}\right)}(X,Y)$$

and

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illustrating first that φ and φ' each have discriminant a square of a rational times that of Φ.

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$$G(X-\alpha Y)(X'-\alpha' Y') = (x-\alpha'' y) =$$

$$(A_x XX' + B_x XY' + C_x X'Y + D_x YY') - \alpha''(A_y XX' + B_y XY' + C_y X'Y + D_y YY'),$$

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$$\begin{pmatrix} A_x & B_x & C_x & D_x \\ A_y & B_y & C_y & D_y \end{pmatrix} = \begin{pmatrix} G & B & C & D \\ 0 & Q/G & Q'/G & -(P+P'+T)/G \end{pmatrix},$$

Now set $\varphi(x,y) = Q(x-\alpha y)(x-\overline{\alpha}y)$, $\varphi'(x,y) = Q'(x-\alpha' y)(x-\overline{\alpha}' y)$. Then $\omega^2 - T\omega + N = 0$, and an intelligent look at the product

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Here B and C are obtained from $BQ' - CQ = G(P-P')$ and the Euclidean algorithm; that yields D, or one obtains it similarly.

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and, this is the point, the "cube law", which sets the compound of these three forms to be the trivial form $(x - \omega y)(x - \overline{\omega} y)$ of their common discriminant, naturally defines a compounding of forms.

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Note that the 'infrastructural composition' I detail is well defined on forms or ideals whereas compounding is well defined only on equivalence classes of forms, or ideals.

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By happy chance, that reduction process also appears in general to find the 'nearest' reduced form (a matter of issue in the real = indefinite case) apparently because that process reduces to the 'previous' reduced form.

The Cubic Case

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Bhargava explains that the correct box is $2 \times 3 \times 3$, thus a pair (A, B) of 3×3 matrices, and that the unique $\mathrm{SL}_3(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$ invariant is a cubic form

$$f(x, y) = ax^3 + bx^2y + cxy^2 + dx^3 = \det(Ax - By)\,.$$

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Hence the unique $\Gamma = \mathrm{GL}_2(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$ invariant is the discriminant $\mathrm{Disc}(f)$ of f.

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Specifically, given a cubic ring R (a ring free of rank 3 as a $\mathbb{Z}$-module), take $(1, \omega, \theta)$ as a Z-basis for R. A 'normal' such basis has $\omega \cdot \theta \in \mathbb{Z}$ and one may define seven integers a, ... ,n by setting

$$\omega\theta = n\,, \qquad \omega^2 = m + b\omega - a\theta\,, \qquad \theta^2 = l + d\omega - c\theta\,.$$

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Then the associative law relations $\omega\theta \cdot \theta = \omega \cdot \theta^2$ and $\omega^2 \cdot \theta = \omega\theta \cdot \theta$ yield as unique solution

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and it follows that a binary cubic form $\det(Ax - By)$ leads to a unique cubic ring. Moreover, a $\mathrm{GL}_2(\mathbb{Z})$ transformation of the basis (ω, θ) of R/Z (and a subsequent renormalisation) transforms $f(x, y)$ by the same transformation.

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The explicit correspondence asks one to write $I = \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{\mathbb{Z}}$, $I' = \langle \alpha'_1, \alpha'_2, \alpha'_3 \rangle_{\mathbb{Z}}$ and, recalling $II' \subseteq R = \langle 1, \omega, \theta \rangle$, to compute all the $\alpha_i \alpha'_j = c_{ij} + a_{ij}\omega + b_{ij}\theta$. Then $A = (a_{ij})$, $B = (b_{ij})$ will do.

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$$(A, B) = \left(\begin{bmatrix} & & 1 \\ & -a & \\ 1 & & -c \end{bmatrix}, \begin{bmatrix} & 1 & \\ 1 & b & \\ & & d \end{bmatrix} \right).$$

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$$-\omega \cdot \alpha_1 = |B_1 A_2 A_3| \cdot \alpha_1 + |A_1 B_1 A_3| \cdot \alpha_2 + |A_1 A_2 B_1| \cdot \alpha_3$$

$$-\omega \cdot \alpha_2 = |B_2 A_2 A_3| \cdot \alpha_1 + |A_1 B_2 A_3| \cdot \alpha_2 + |A_1 A_2 B_2| \cdot \alpha_3$$

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In the case of binary quadratic forms, the unique $\mathrm{SL}_2(\mathbb{Z})$-invariant is the discriminant D, which classifies orders in quadratic fields. The primitive classes having a fixed value of D form a group under a certain natural composition law. This group is naturally isomorphic to the narrow class group of the corresponding quadratic order.

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In the case of $2 \times 3 \times 3$ integer boxes, the unique $\mathrm{SL}_3(\mathbb{Z}) \times \mathrm{SL}_3(\mathbb{Z})$-invariant is the cubic form f, which classifies orders in cubic fields. The projective classes having a fixed value of f form a group under a certain natural composition law. This group is naturally isomorphic to the ideal class group of the corresponding cubic order.

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